ECE 635 – 11.10.2011 (1)

Derivation of ray path in step index and graded index fibres

s measures distance

Fermat's Principle

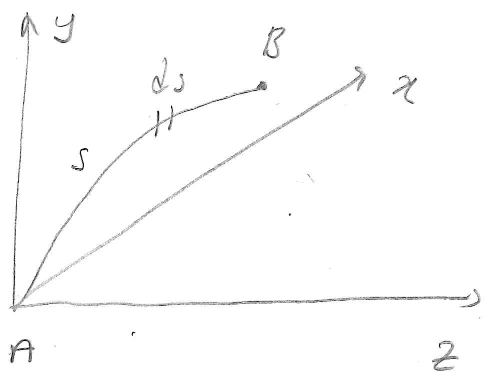
vectorial

$\vec{R} = (x, y, z)$

$$\delta \int_A^B n(R)\, ds = 0$$

δ = Variation of

ds = differential length along the ray trajectory between A and B defined by $x(s)$, $y(s)$, $z(s)$ as shown below



From this expression, we get three partial differential equations that must be satisfied by $x(s)$, $y(s)$, $z(s)$

$$\frac{d}{ds}\left(n\frac{dx}{ds}\right) = \frac{\partial n}{\partial x}, \quad \frac{d}{ds}\left(n\frac{dy}{ds}\right) = \frac{\partial n}{\partial y}$$

$$\frac{d}{ds}\left(n\frac{dz}{ds}\right) = \frac{\partial n}{\partial z}$$

By defining $R = [x(s), y(s), z(s)]$

$$\frac{d}{ds}\left(n\frac{dR}{ds}\right) = \nabla n \rightarrow \text{compact form of ray equation}$$

where $\nabla n = \frac{\partial n}{\partial x} + \frac{\partial n}{\partial y} + \frac{\partial n}{\partial z}$, $n \rightarrow$ refractive index of medium

$ds^2 = dx^2 + dy^2 + dz^2$ or

$$ds = dz\sqrt{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2}$$

If ray travels almost parallel to z axis, then

$s \simeq z$ or $ds \simeq dz$ and the third DE

$$\frac{d}{ds}\left(n\overset{1}{\frac{dz}{ds}}\right) = \frac{\partial n}{\partial z} \quad \text{will become}$$

$$\frac{d}{dz}(n) \simeq \frac{dn}{dz}$$ so this is redundant, hence

we have only two remaining equations

$$\frac{d}{dz}\left(n\,\frac{dx}{dz}\right) \simeq \frac{\partial n}{\partial x}\,, \quad \frac{d}{dz}\left(n\,\frac{dy}{dz}\right) \simeq \frac{\partial n}{\partial y}$$

Given $n = n(x,y,z)$ these two partial DEs

can be solved to get $x(z)$ and $y(z)$

If the medium is homogeneous i.e. $n(x,y,z) = n$

no x, y, z variation then

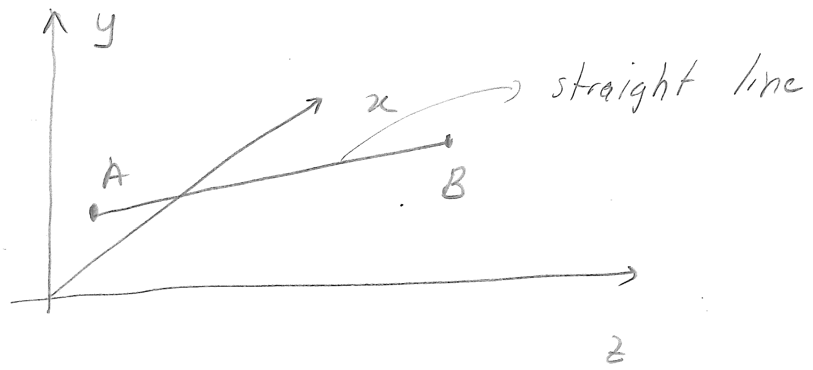
$$\frac{d^2x}{dz^2} = 0 \quad \text{and} \quad \frac{d^2y}{dz^2} = 0$$

In this case solutions are

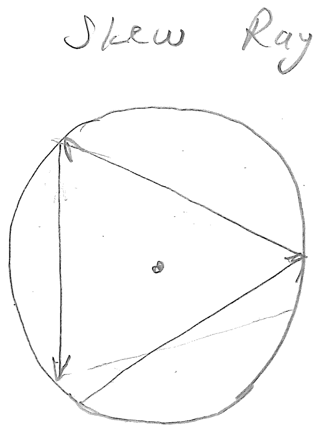
$$x(z) = A_x z + A_{x0}\,, \quad y(z) = A_y z + A_{y0}$$

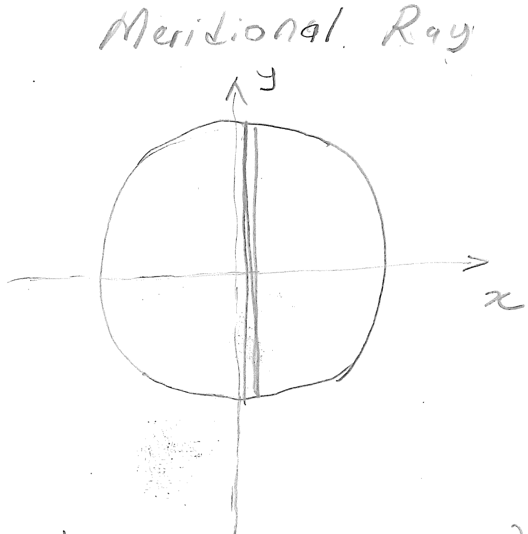
This means that ray trajectories are

Straight lines, A_x, A_{x0}, A_y, A_{y0} are to be determined from boundary conditions (or initial conditions)



This is the condition in step index fibres as well and the end having a cross section we view the fibre it is seen as





A ray whose $x(z), y(z) \neq 0$

A ray whose $x(z) = 0$, $y(z) \neq 0$

It is more interesting to investigate the ray propagation in graded index fibres in this case (with parabolic profile)

$$n(r) = n_1 \left[1 - 2\Delta (r/a)^2\right]^{1/2}$$

$$= n_1 \left[1 - 2\Delta r_n^2\right]^{1/2} \quad \text{or in } x, y$$

$$n(x,y) = n_1 \left[1 - 2\Delta (x^2+y^2)/a^2\right]^{1/2}$$

$$= n_1 \left[1 - 2\Delta (x_n^2 + y_n^2)\right]^{1/2}$$

If Δ is small which is the case in graded index fibres, then

$n(x) \simeq n_1 (1 - \Delta x_n^2)$, insert this into paraxial equation (for x)

$$\frac{d}{dz}\left[n_1 (1 - \Delta x_n^2) \frac{dx}{dz}\right] \simeq \frac{dn}{dx}$$

Setting $1-\Delta x_n^2 \simeq 1$ (since Δ is too small)

$$\frac{d^2x}{dz^2} = -\frac{2n_1\Delta}{n_1(1-\Delta x_n^2)}\frac{x}{a^2} \simeq -\frac{2\Delta}{a^2}x$$

Similarly we find for y

$$\frac{d^2y}{dz^2} = -\frac{2\Delta}{a^2}y$$, solution to these DEs are

harmonic functions i.e which are

$$x(z) = A_x \cos\left(\frac{\sqrt{2\Delta}}{a}z\right) + B_x \sin\left(\frac{\sqrt{2\Delta}}{a}z\right)$$

$$y(z) = A_y \cos\left(\frac{\sqrt{2\Delta}}{a}z\right) + B_y \sin\left(\frac{\sqrt{2\Delta}}{a}z\right)$$

By imposing initial conditions that at $z=0$

$$x(z=0) = x_0 , \quad y(z=0) = y_0$$

$$\frac{d}{dz}x(=0) = \theta_{x0}$$ (tangent of the angle) $$\frac{d}{dz}y(=0) = \theta_{y0}$$

$$x(z) = x_0 \cos\left(\frac{\sqrt{2\Delta}}{a} z\right) + \frac{a\,\theta_{x0}}{\sqrt{2\Delta}} \sin\left(\frac{\sqrt{2\Delta}}{a} z\right)$$

$$y(z) = y_0 \cos\left(\frac{\sqrt{2\Delta}}{a} z\right) + \frac{a\,\theta_{y0}}{\sqrt{2\Delta}} \sin\left(\frac{\sqrt{2\Delta}}{a} z\right)$$

The above is plotted in Ray-tracing-GI-Exp2.m in ECE 474 folder of MATLAB.

To find r_{min}, r_{max}, we must write for $r(z)$ such that

$$r(z) = [x^2(z) + y^2(z)]^{1/2}$$

$$r^2(z) = (x_0^2 + y_0^2) \cos^2\left(\frac{\sqrt{2\Delta}}{a} z\right)$$

$$+ \frac{a^2}{2\Delta}(\theta_{x0}^2 + \theta_{y0}^2) \sin^2\left(\frac{\sqrt{2\Delta}}{a} z\right)$$

$$\frac{x_0\, a\, \theta_{x0}}{\sqrt{2\Delta}} \sin\left(\frac{2\sqrt{2\Delta}}{a} z\right) + \frac{y_0\, a\, \theta_{y0}}{\sqrt{2\Delta}} \sin\left(\frac{2\sqrt{2\Delta}}{a} z\right)$$

Now we differentiate $r(z)$ wrt z

and take the derivative of $\frac{d}{dz}\left[x^2(z)+y^2(z)\right]$

since the denominator will be multiplied by zero

$$-\frac{\sqrt{2D}}{a}\left(x_0^2+y_0^2\right)\sin\left(\frac{2\sqrt{2D}}{a}z\right)$$

$$+\frac{a}{\sqrt{2D}}\left(\theta_{x_0}^2+\theta_{y_0}^2\right)\sin\left(\frac{2\sqrt{2D}}{a}z\right)$$

$$+2x_0\theta_{x_0}\cos\left(\frac{2\sqrt{2D}}{a}z\right)+2y_0\theta_{y_0}\cos\left(\frac{2\sqrt{2D}}{a}z\right)=0$$

$$\frac{\sin\left(\frac{2\sqrt{2D}}{a}z\right)}{\cos\left(\frac{2\sqrt{2D}}{a}z\right)}=\tan\left(\frac{2\sqrt{2D}}{a}z\right)$$

$$=\frac{2\left(x_0\theta_{x_0}+y_0\theta_{y_0}\right)}{\frac{\sqrt{2D}}{a}\left(x_0^2+y_0^2\right)-\frac{a}{\sqrt{2D}}\left(\theta_{x_0}^2+\theta_{y_0}^2\right)}$$

$$z_p = \frac{a}{2\sqrt{2D}} \; a\tan\left[\qquad \right]$$

$$r_{min}(z) = \left[x^2(z_p) + y^2(z_p) \right]^{1/2}$$

If we take $a\tan[\qquad] + \pi$, we find

$$z_{p(max)} = \frac{a}{2\sqrt{2D}} \left\{ a\tan[\qquad] + \pi \right\} \quad \text{and}$$

$$r_{max}(z) = \left[x^2(z_{p\,max}) + y^2(z_{p\,max}) \right]^{1/2}$$

The above works ok in Ray-tracing-GL-Exp2.m

in ECE 474 folder of MATLAB.

Eikonal Equation

This is another form of Fermat's principle which states that if $S(R)$ represents the the phase front surface, then

$|\nabla S|^2 = n^2$ where ∇ is the gradient operator

Now in graded index fibre, we consider that a quasi plane wave travels, which is shown as

$$E(R) = e(R) \exp[-jk\, S(R)] \qquad \begin{matrix} R = (x, y, z) \\ R = (r, \phi, z) \end{matrix}$$

Here $e(R)$ and $S(R)$ vary slowly in comparison with wavelength λ. From above we know that S is also related to refractive index from our previous developments we know that

dependence on ϕ is as $\exp(-j\nu\phi)$

and dependence on z is as $\exp(-j\beta z)$

on r as $k_r(r)$

finally we formulate the dependence on r

$\exp\left(-j\int_0^r k_r(r)\,dr\right)$, hence

$-jkS(R) = -j\int_0^r k_r(r)\,dr - j\nu\phi - j\beta z$

Now apply $\nabla = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial z}$ and

set $k^2|\nabla S(R)|^2 = n^2(r)k^2$

$$k_r^2(r) = n^2(r)k^2 - \beta^2 - \frac{\nu^2}{r^2}$$

The wave travels within the core where

k_r^2 is positive

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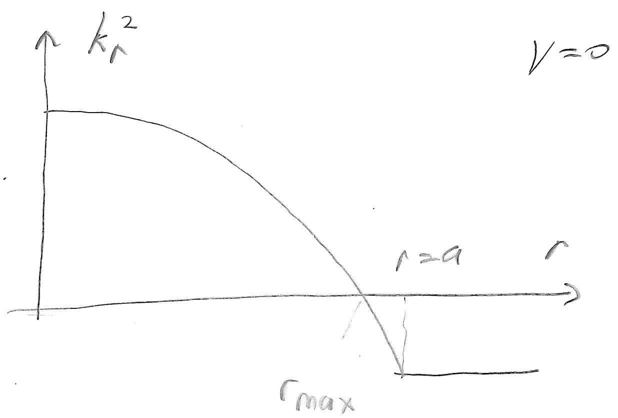
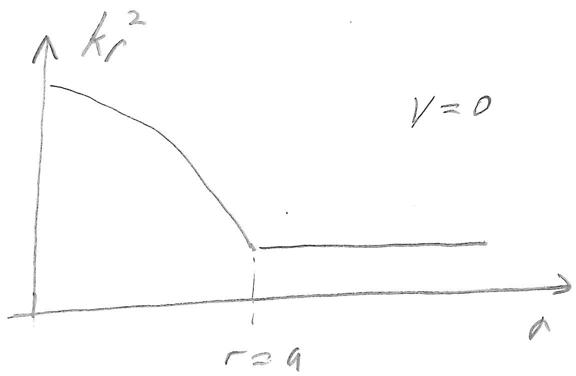
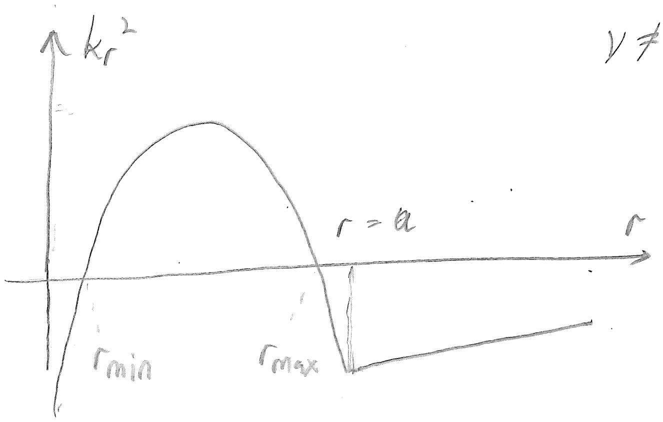
outside these regions k_r^2 is negative or k_r is imaginary, so the wave decay exponentially there, In general we have the following picture



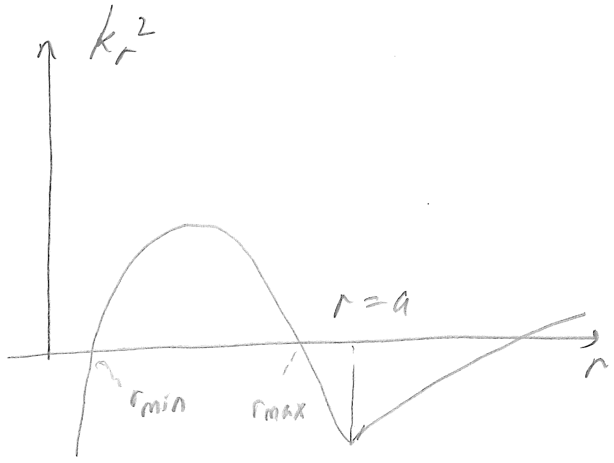
$\Rightarrow$ Meridional Bound (Reflecting) Guided



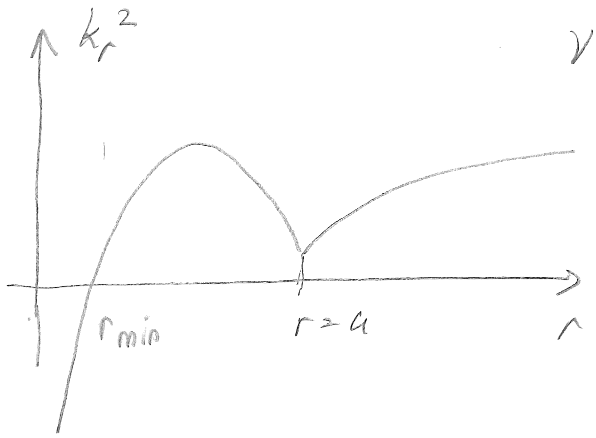
$\Rightarrow$ Meridional Refracting



Skew Guided Bound



$\nu \neq 0$ Skew Refracting (Leaky)



$\nu \neq 0$ Skew refracting

On the other hand, by taking the integral of k_r between r_{min} and r_{max} (which are found by setting $k_r = 0$ and retrieving roots)

$$2 \int_{r_{min}}^{r_{max}} k_r \, dr = 2\pi m \quad , \ m \rightarrow \text{integer}$$

which means that radial path length must correspond to a phase shift of $m \times 2\pi$

Solving this integral will give us for

$n(r) = n_1 \left[1 - 2\Delta (r/a)^2 \right]^{1/2}$, i.e. quadratic profile

$$\frac{a(k^2 n_1^2 - \beta^2)}{4 k n_1 \sqrt{2\Delta}} - \frac{V}{2} = m$$

From here it is possible to find β's in a simple manner given m and such that

$$\beta = [kn_1^2 - 4kn_1\sqrt{2\Delta}(m+\nu/2)/a]^{1/2}$$

From this equation we can determine β (propagation constant) of graded index fibres and also check which β will correspond to which class of ray listed on pp. 12, 13