ECE 572 - HTE - 9.12.2009

Dispersion in Single Mode fibres (1)

Since only one mode exists dispersion is entirely due to source (Laser, LED) containing many spectral components (waveguide dispersion) thus exciting a different HE_{11} mode (or causing fibre to have a different V) and refractive index changes of fibre with wavelength. (material dispersion)

(books) ch. 2
Based on GP Agrawal, Fibre Optic Communication Systems and Saleh & Teich, Fundamentals of Photonics ch. 5

Derivation of Group Velocity
Initially we wish to set some fundamentals.

A plane wave in free space has at r=0 point (monochromatic case)

$$U(r=0,t) = A \exp(j 2\pi f_1 t)$$

(in free space)
As it travels a distance z (if paraxial approximation is valid $r \approx z$) a time of $\frac{z}{c}$ passes where c = speed of light in (also phase velocity in this case) free space, it becomes

This means replace t (at the source) with $t - z/c$ ↓

$$u[r = (x, y, z), t] = A \exp\left[j 2\pi f_1 \left(t - \frac{z}{c}\right)\right]$$

$$= A \exp(j 2\pi f_1 t) \exp\left(-j \overbrace{2\pi f_1}^{\omega_1} \frac{z}{c}\right)$$

$$= A \exp(j 2\pi f_1 t) \exp(-j k z) \quad \text{with} \quad k = \frac{\omega_1}{c} = \frac{2\pi}{\lambda}$$

This way time (lapse) and space is combined into k

Now consider a pulsed waveform of

$$u(0, t) = A(t) \exp(j 2\pi f_1 t)$$

This pulse can no longer be monochromatic, but will occupy a frequency band around f_0 depending on the form of $A(t)$

Alternatively this may be considered to represent a polychromatic source with a central frequency of f_1.

Now we wish to find $u(r,t)$

This can be done simply by replacing t by $t - z/c$ as above, so for a polychromatic source

$$u(r,t) = A\left(t - \frac{z}{c}\right) \exp\left[j 2\pi f_1 \left(t - \frac{z}{c}\right)\right]$$

$$= A\left(t - \frac{z}{c}\right) \exp\left(j 2\pi f_1 t\right) \exp\left(-j k z\right)$$

Now the above is valid for propagation in free space. When a polychromatic source is launched into a medium with refractive index n, then simply do the following replacement in the above formulations

$$c \rightarrow \underset{\substack{\downarrow \\ \text{new phase velocity}}}{v} = \frac{c}{n} \qquad \text{thus} \qquad k \rightarrow k_1 = \frac{\omega_1}{c} n \rightarrow \beta$$

In this case the field at z will be

$$U(r,t) = A\left(t - \underbrace{\frac{zn}{c}}\right) \exp\left(j2\pi f_0 t\right) \exp\left(-j\beta z\right)$$

indicates that there is more time delay in a medium of $n > 1$ (practically the case always)

Note also that in a propagating medium other than free space it is customary to show the propagation constant as β instead of k (even in turbulence)

Now represent the polychromatic source as the sum of two complex exponentials having the same amplitude (take f_1 and f_2, $f_1 \to$ for monochromatic case)

$$U(r=0,t) = A\left[\exp\left(j2\pi f_1 t\right) + \exp\left(j2\pi f_2 t\right)\right]$$

In this case propagation delay should be reflected directly into exponentials

This will result in AM, to see this consider only the real (or imaginary part), thus

$$U_r(r=0,t) = A\left[\cos(2\pi f_1 t) + \cos(2\pi f_2 t)\right]$$

$$= 2A \cos\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]\cos\left[2\pi\left(\frac{f_1+f_2}{2}\right)\right]$$

This constitues DSB so the envelope will be shaped by $\cos\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]$

Similar to the derivations shown on pp. 2 we can consider each exponential component at f_1 and f_2 to propagate individually so

$$U(r=z,t) = A\left[\exp(j2\pi f_1 t - jk_1 z) + \exp(j2\pi f_2 t - jk_2 z)\right]$$

note

$$k_1 = \frac{\omega_1}{c} n(\lambda)$$

$$k_2 = \frac{\omega_2}{c} n(\lambda)$$

Again real part of this may be considered to constitute AM again we consider the real part

$$U_r(r=z,t) = 2A \cos\left[2\pi \frac{(f_1-f_2)}{2} t - (k_1-k_2) z\right]$$

$$\cos\left[2\pi \frac{(f_1+f_2)}{2} t - (k_1+k_2) z\right]$$

Again the first term corresponds to the envelope and in analogy to the definition on pp. 3 the equivalent k_{eq} will be

(pp. 3 def.)

$$k = \frac{\omega_1}{c} \quad , \quad k_{eq} = \frac{2\pi(f_1-f_2)}{c_{eq}} = k_1 - k_2$$

$$c_{eq} \text{ or } V_g = \frac{2\pi(f_1-f_2)}{k_1-k_2} = \frac{\omega_1-\omega_2}{k_1-k_2}$$

now if we are in a dispersive media which means $\overbrace{n(\lambda) \text{ or } n(f)}^{\text{source of material dispersion}}$ or

different wavelength at each n ← $\lambda(n)$ or $f(n)$ or $\omega(n)$

or $\omega(k)$

$$V_g = \frac{\omega_1(k_1) - \omega_2(k_2)}{k_1 - k_2}$$

expand the above at $k = 0.5(k_1 + k_2)$

thus $k_1 = 2k - k_2$, $k_2 = 2k - k_1$

Expand the above in a Tylor series (retain first two terms only)

$$V_g = \frac{\omega(2k - k_2) - \omega(2k - k_1)}{k_1 - k_2}$$

$$= \frac{\omega(2k) - k_2\,\omega'(2k) \cdots \quad - \omega(2k) + k_1\,\omega'(2k)}{k_1 - k_2}$$

$$\simeq \omega'(2k) = \frac{d\omega}{dk} = V_g$$

In a fibre propagating medium replace k with β , thus in fibre

$$V_g = \frac{d\omega}{d\beta} \quad \text{or} \quad V_g = \left(\frac{d\beta}{d\omega}\right)^{-1}$$

Now we continue with pp. 38 of Agrawal

(mostly approximated to n_1 or n_2)

From $\bar{n} = \dfrac{\beta}{k}$ where $\bar{n}$ is mode index

$(k = 2\pi/\lambda)$

$\beta = \bar{n}k = \bar{n}\dfrac{w}{c}$ (where c = speed of light in free space)

$$\frac{d\beta}{dw} = \frac{\bar{n}}{c} + w\frac{d\bar{n}}{dw}\frac{1}{c}$$

So $$V_g = \left(\frac{d\beta}{dw}\right)^{-1} = \frac{c}{\bar{n} + w\dfrac{d\bar{n}}{dw}}$$ [Eq. 2.3.2 of Agr]

$(T = L/V_g)$

Now T is the time taken for the wave to traverse the fibre or a pulse with duration of T is transmitted and this is broadened by ΔT and source (or the pulse) has a spectral width of σ_w (as measured in radians)

$$\Delta T = \frac{dT}{dw}\sigma_w = \frac{d}{dw}\left(\frac{L}{V_g}\right)\sigma_w$$

$$\Delta T = L \frac{d^2\beta}{d\omega^2} \sigma_\omega = L \beta_2 \sigma_\omega$$

$\beta_2 = \frac{d^2\beta}{d\omega^2}$ is Group Velocity dispersion (GVD) of fibre

We can also formulate the above in terms of λ (instead of ω). To change to λ apply the following

$$\overset{d\omega}{\sigma_\omega} = 2\pi \overset{df}{\sigma_f} = -2\pi \frac{c}{\lambda^2} \overset{d\lambda}{\sigma_\lambda} \qquad \left(\text{from } \omega = 2\pi f = \frac{2\pi c}{\lambda}\right)$$

source spectral width [σ_λ for ELE 474]

will cancel

$$\frac{d\omega}{d\lambda} = -\frac{2\pi c}{\lambda^2} \quad \text{or} \quad \frac{d}{d\omega} = -\frac{\lambda^2}{2\pi c} \frac{d}{d\lambda}, \text{ thus}$$

use this

$$\frac{d}{d\omega}\left(\frac{L}{V_g}\right) \sigma_\omega = \frac{d}{d\lambda}\left(\frac{L}{V_g}\right) \sigma_\lambda = D L \sigma_\lambda$$

where $D = \frac{d}{d\lambda}\left(\frac{1}{V_g}\right) = -\frac{2\pi c}{\lambda^2} \beta_2$ [Eq 2.3.5 of Agr]

D is called the dispersion parameter. Another definition is

$$\Delta T \doteq L \beta_2 \sigma_\omega = L\left(-D \frac{\lambda^2}{2\pi c}\right)\left(-2\pi \frac{c}{\lambda^2} \sigma_\lambda\right) = L D \sigma_\lambda$$

Similar to multimode fibre, we desire that

$$B\,\overbrace{L\,|D|\,\sigma_\lambda}^{\Delta T} < 1 \quad \text{or} \quad B \cdot \Delta T < 1$$

Bandwidth offered by the fibre

We can split up the parameter D as follows

$$D = -\frac{2\pi c}{\lambda^2}\frac{d}{d\omega}\left(\frac{1}{V_g}\right) = -\frac{2\pi c}{\lambda^2}\frac{d^2\beta}{d\omega^2} \qquad \beta \to \bar{n}\omega/c$$

$$= -\frac{2\pi}{\lambda^2}\left(2\frac{d\bar{n}}{d\omega} + \omega\frac{d^2\bar{n}}{d\omega^2}\right) = \underbrace{D_m}_{\text{Material}} + \overbrace{D_w}^{\text{waveguide}}$$

Now substitute from Eq. 2.2.39 of Agr

$$\bar{n} \simeq n_2(1 + b\Delta), \quad \text{where } b \overset{\text{approximation}}{\simeq} (1.1428 - 0.996/V)^2$$

and $V \simeq \frac{2\pi}{\lambda} a n_1 \sqrt{2\Delta}$

Considering $\bar{n} = \overbrace{n_2}^{\text{will give } D_m} + \underbrace{n_2 b \Delta}_{\text{will give } D_w}$

For n_2 only

$$2\frac{d\bar{n}}{d\omega} + \omega\frac{d^2\bar{n}}{d\omega^2} = 2\frac{dn_2}{d\omega} + \omega\frac{d^2 n_2}{d\omega^2}$$

Definition of extra parameters

Define

$$b = \frac{\beta/k - n_1}{n_1 - n_2} = \text{(Alternative) Normalized Propagation constant}$$

Now call $\beta/k = \bar{n}$ mode index

$$b = \frac{\bar{n} - n_2}{n_1 - n_2} \quad , \quad 2\Delta = \frac{n_1^2 - n_2^2}{n_1^2} = \frac{(n_1 - n_2)\overbrace{(n_1 + n_2)}^{2n_1}}{n_1^2}$$

$$= \frac{n_1 - n_2}{n_1^2} \times 2n_1$$

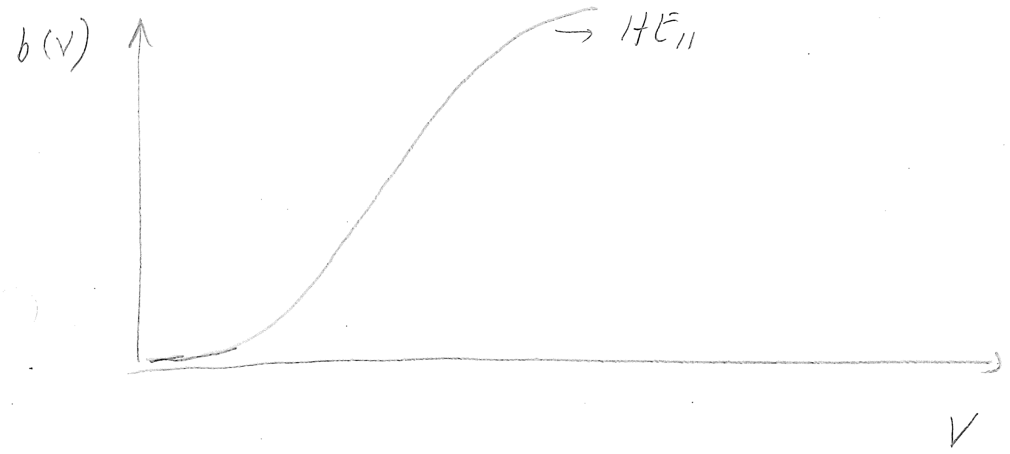
$$\Delta \simeq \frac{n_1 - n_2}{n_1} \quad , \quad n_1 - n_2 = \Delta n_1$$

$$\bar{n} - n_2 = \Delta n_1 b \quad \text{or} \quad \bar{n} = n_2 + \Delta n_1 b$$

or in terms of n_2 purely $\quad \bar{n} = n_2 + \Delta n_2 b$

$$= n_2 (1 + \Delta b)$$

When β is calculated from CE and plotted against V we get



This is of course difficult to describe with a simplified expression, instead

$$b(V) = (1.1428 - 0.9960/V)^2$$

for V in the range $1.5 < V < 2.5$

$$= \frac{d}{d\omega}\Big(\overbrace{n_2 + \omega \frac{dn_2}{d\omega}}^{n_{2g}}\Big) = \frac{d}{d\omega} n_{2g}, \quad \text{thus}$$

$$D_M = -\frac{2\pi}{\lambda^2}\Big(\overbrace{2\frac{d\bar{n}}{d\omega} + \omega \frac{d^2\bar{n}}{d\omega^2}}^{n_2 \text{ part only}}\Big)$$

$$= -\frac{2\pi}{\lambda^2}\frac{d}{d\omega} n_{2g} = \frac{1}{c}\frac{d}{d\lambda} n_{2g} = -\frac{\lambda}{c}\frac{d^2 n_2}{d\lambda^2}$$

For the D_W part on the other hand

$$D_W = -\frac{2\pi}{\lambda^2}\left[2\frac{d(n_2 b\Delta)}{d\omega} + \omega \frac{d^2(n_2 b\Delta)}{d\omega^2}\right]$$

$$= -\frac{2\pi\Delta}{\lambda^2}\left[2b\frac{dn_2}{d\omega} + 2n_2\frac{db}{d\omega} + \omega\frac{d}{d\omega}\left(b\frac{dn_2}{d\omega} + n_2\frac{db}{d\omega}\right)\right]$$

convert $\frac{db}{d\omega}$ to $\frac{db}{dV}$ via

Use Eq. (2.3.10) of Agr

$$V = k a n_1 \sqrt{2\Delta} = \frac{\omega}{c} a n_1 \sqrt{2\Delta}$$

$$\frac{1}{d\omega} = \frac{1}{c} a n_1 \sqrt{2\Delta} \frac{d}{dV} = \frac{k}{\omega} a n_1 \sqrt{2\Delta} \frac{d}{dV}$$

$$= \frac{V}{\omega} \frac{d}{dV}$$, substitute this into above to get

$$D_\omega = -\frac{2\pi\Delta}{\lambda^2} \left\{ 2b \frac{dn_2}{d\omega} + \frac{2n_2}{\omega} V \frac{db}{dV} + \omega \overbrace{\frac{d}{d\omega}}^{\frac{V}{\omega}\frac{d}{dV}} \left[b \frac{dn_2}{d\omega} + \frac{n_2}{\omega} V \frac{db}{dV} \right] \right\}$$, after rearrangement

$$= -\frac{2\pi\Delta}{\lambda^2} \left[\frac{n_{2g}^2}{n_2 \omega} \frac{V d^2(Vb)}{dV^2} + \frac{dn_{2g}}{d\omega} \frac{d(Vb)}{dV} \right]$$

So to calculate D_M, we need the graphs of

n_2, $\frac{d^2 n_2}{d\lambda^2}$

To calculate D_ω on the other hand we need

graphs of $\frac{d(Vb)}{dV}$ and $\frac{d^2(Vb)}{dV^2}$

(against V)

Refer to Eq. (2.3.10) of Agrawal